Unit II Maneuvering Flight Performance

2.1. Range

Range is the total distance traversed by an airplane on one load of fuel.

Consider the following weights:

W_0- gross weight of the airplane including everything; full fuel load, payload, crew, structure, etc.

W_f- weight of fuel; this is an instantaneous value, and it changes as fuel is consumed during flight.

W_e- weight of the airplane when the fuel tanks are empty.

At any instant during the flight, the weight of the airplane W is,

$$W = W_e + W_f$$

Since W_f is decreasing during the flight, W is also decreasing. Indeed, the time rate of change of weight is

$$\frac{dW}{dt} = -\frac{dW_f}{dt} = -\dot{W}_f$$

The above equation is negative because fuel is being consumed. Range is intimately connected with engine performance through the specific fuel consumption for a propeller-driven / reciprocating engine.

$$c = -\frac{\dot{W}_f}{P}$$

where, $P \rightarrow$ shaft power.

And the thrust specific fuel consumption for a jet-propelled airplane is,

$$c_t = -\frac{\dot{W}_f}{T}$$

where, $T \rightarrow$ available thrust.

Now $c_t$can be expressed in terms of c and vice versa. So,

$$c_t = \frac{cV_\infty}{\eta_{pr}}$$

where, $\eta_{pr} \rightarrow$ propeller efficiency.

Above equation is particularly used for generating an equivalent "thrust" specific fuel consumption for propeller-driven airplanes.

Consider an airplane in steady, level flight. Let s denote horizontal distance covered over the ground. The airplane's velocity V_∞ is,

$$V_\infty = \frac{ds}{dt}$$

$$ds = V_\infty dt$$

The thrust specific fuel consumption for a jet-propelled airplane can be written as

$$c_t = -\frac{\dot{W}_f}{T} = -\frac{dW_f}{dt}\frac{1}{T}$$

$$dt = -\frac{dW_f}{c_t T}$$

Then the ds will be as follows. Where, $dW_f = dW$

$$ds = -\frac{V_\infty}{c_t T} dW_f = -\frac{V_\infty}{c_t T} dW$$

$$ds = -\frac{V_\infty}{c_t} \frac{W}{T} \frac{dW}{W}$$

In steady, level flight, $L = W$ and $T = D$.

$$ds = -\frac{V_\infty}{c_t} \frac{L}{D} \frac{dW}{W}$$

Integrating above from $s = 0$ where $W = W_0$ to $s = R$ where $W = W_e$ will give

$$\int_0^R ds = -\int_{W_0}^{W_e} \frac{V_\infty}{c_t} \frac{L}{D} \frac{dW}{W}$$

$$= \frac{V_\infty}{c_t} \frac{L}{D} \int_{W_e}^{W_0} \frac{dW}{W}$$

$$R = \frac{V_\infty}{c_t} \frac{L}{D} \ln \frac{W_0}{W_e}$$

This equation is called the *Breguet range equation*. So the range is influenced by,

1. Lift to drag ratio.
2. Specific fuel consumption
3. Velocity
4. Initial amount of fuel.

2.1.1. *Range for Propeller-Driven Airplanes*

The *Breguet range equation* is

$$R = \frac{V_\infty}{c_t} \frac{L}{D} \ln \frac{W_0}{W_e}$$

We know that c_t in terms of c is,

$$c_t = \frac{c V_\infty}{\eta_{pr}}$$

Then,

$$R = \frac{\eta_{pr}}{c} \frac{L}{D} \ln \frac{W_0}{W_e}$$

This is the historical Breguet range equation. For maximum range, the propeller-driven / reciprocating engine power plant should;

1. Fly at maximum L/D.
2. Have the highest possible propeller efficiency.
3. Have the lowest possible specific fuel consumption.
4. Have the highest possible ratio of W_0 to W_e.

2.1.2. Range for Jet-Propelled Airplanes

The Breguet range equation is the range equation for a jet-propelled airplane. The maximum value of the range is obtained from the product $V_\infty(L/D)$. Let us examine this product for steady, level flight,

$$L = W = \frac{1}{2}\rho_\infty V_\infty^2 S C_L$$

$$V_\infty = \sqrt{\frac{2W}{\rho_\infty S C_L}}$$

Thus,

$$V_\infty \frac{L}{D} = \sqrt{\frac{2W}{\rho_\infty S C_L}}\frac{C_L}{C_D} = \sqrt{\frac{2W}{\rho_\infty S}}\frac{C_L^{1/2}}{C_D}$$

Thus the product $V_\infty(L/D)$ is maximum when the airplane is flying at a maximum value of $C_L^{1/2}/C_D$. Now

$$R = \frac{1}{c_t}\sqrt{\frac{2W}{\rho_\infty S}}\frac{C_L^{1/2}}{C_D}\int_{W_e}^{W_0}\frac{dW}{W}$$

Assuming of constant c_t, ρ_∞, S, and $C_L^{1/2}/C_D$ are constant.

$$R = \frac{1}{c_t}\sqrt{\frac{2}{\rho_\infty S}}\frac{C_L^{1/2}}{C_D}\int_{W_e}^{W_0}\frac{dW}{W^{1/2}}$$

$$R = \frac{2}{c_t}\sqrt{\frac{2}{\rho_\infty S}}\frac{C_L^{1/2}}{C_D}\left(W_0^{1/2} - W_e^{1/2}\right)$$

From this equation, the flight conditions for maximum range for a jet-propelled airplane are

1. Fly at maximum $C_L^{1/2}/C_D$.
2. Have the lowest possible thrust specific fuel consumption.
3. Have the highest possible difference between W_0 and W_e.
4. Fly at high altitude, where ρ_∞ is small.

2.2. Endurance

Endurance is the amount of time that an airplane can stay in the air on one load of fuel. We know that,

$$c_t = -\frac{\dot{W}_f}{T} = -\frac{dW_f}{dt}\frac{1}{T}$$

$$\frac{dW_f}{dt} = -c_t T$$

$$dt = -\frac{dW_f}{c_t T}$$

Since $T = D$ and $L = W$ in steady, level flight, the above equation can be written as

$$dt = -\frac{dW_f}{c_t D} = -\frac{1}{c_t}\frac{L}{D}\frac{dW_f}{W}$$

Integrating above from $t = 0$ where $W = W_0$ to $t = E$ where $W = W_e$ will give

$$\int_0^E dt = -\int_{W_0}^{W_e} \frac{1}{c_t}\frac{L}{D}\frac{dW_f}{W}$$

$$E = \int_{W_e}^{W_0} \frac{1}{c_t}\frac{L}{D}\frac{dW_f}{W}$$

Above equation is the general equation for the endurance E of an airplane.

2.2.1. Endurance for Propeller-Driven Airplanes

The specific fuel consumption for propeller-driven airplanes is given in terms of power rather than thrust. The relation between c and c_t is

$$c_t = \frac{cV_\infty}{\eta_{pr}}$$

Substituting this relation in endurance equation will give

$$E = \int_{W_e}^{W_0} \frac{\eta_{pr}}{cV_\infty}\frac{C_L}{C_D}\frac{dW_f}{W}$$

$$E = \int_{W_e}^{W_0} \frac{\eta_{pr}}{c}\sqrt{\frac{\rho_\infty S C_L}{2W}}\frac{C_L}{C_D}\frac{dW_f}{W}$$

$$E = \int_{W_e}^{W_0} \frac{\eta_{pr}}{c}\sqrt{\frac{\rho_\infty S}{2}}\frac{C_L^{3/2}}{C_D}\frac{dW_f}{W^{3/2}}$$

By making the assumptions of constant η_{pr}, c, ρ_∞, and $C_L^{3/2}/C_D$

$$E = \frac{\eta_{pr}}{c}\sqrt{\frac{\rho_\infty S}{2}}\frac{C_L^{3/2}}{C_D}\int_{W_e}^{W_0}\frac{dW_f}{W^{3/2}}$$

$$E = \frac{\eta_{pr}}{c}\sqrt{\frac{\rho_\infty S}{2}}\frac{C_L^{3/2}}{C_D}\int_{W_e}^{W_0} W^{-3/2} dW_f$$

$$E = \frac{\eta_{pr}}{c}\sqrt{\frac{\rho_\infty S}{2}}\frac{C_L^{3/2}}{C_D}\frac{\left[W^{-1/2}\right]_{W_e}^{W_0}}{-1/2}$$

$$E = \frac{\eta_{pr}}{c}\sqrt{2\rho_\infty S}\frac{C_L^{3/2}}{C_D}\left(W_e^{-1/2} - W_0^{-1/2}\right)$$

The maximum endurance for a propeller-driven airplane corresponds to the following conditions.

1. Fly at maximum $C_L^{3/2}/C_D$.
2. Have the highest possible propeller efficiency.

3. Have the lowest possible specific fuel consumption.
4. Have the highest possible difference between W_0 and W_e.
5. Fly at sea level, where ρ_∞ is the largest value.

2.2.2. *Endurance for Jet-Propelled Airplanes*

If we assume flight at constant c_t and L/D. The equation in terms of thrust specific fuel consumption is the endurance for a jet-propelled airplane.

$$E = \frac{1}{c_t}\frac{L}{D}\int_{W_e}^{W_0}\frac{dW_f}{W}$$

$$E = \frac{1}{c_t}\frac{L}{D}\ln\frac{W_0}{W_e}$$

The maximum endurance for a jet-propelled airplane corresponds to the following conditions:

1. Fly at maximum L/D.
2. Have the lowest possible thrust specific fuel consumption.
3. Have the highest possible ratio of W_0 to W_e.

2.3. Rate of Climb

Focus to an airplane in steady, unaccelerated climbing flight. The climb angle θ defined as the angle between the instantaneous flight path direction and the horizontal.

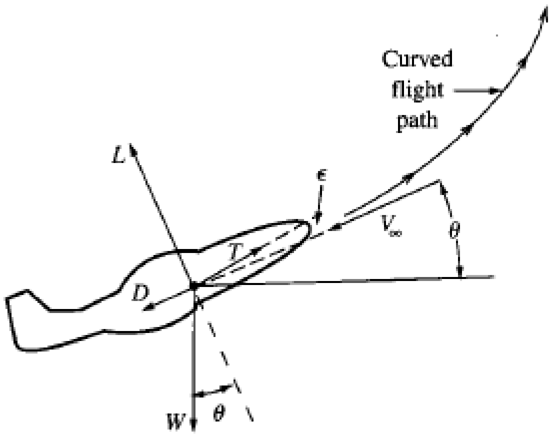


Fig. 2.1. Climbing flight.

The equations of motion for accelerated flight along a curved flight path are given by,

$$m\frac{dV_\infty}{dt} = T\cos\epsilon - D - W\sin\theta$$

$$m\frac{V_\infty^2}{r_1} = L\cos\emptyset + T\sin\epsilon\cos\emptyset - W\cos\theta$$

$$m\frac{(V_\infty\cos\theta)^2}{r_2} = L\sin\emptyset + T\sin\epsilon\sin\emptyset$$

The equations of motion for steady, unaccelerated climbing flight along a straight path are given by,

$$T\cos\epsilon - D - W\sin\theta = 0$$
$$L + T\sin\epsilon - W\cos\theta = 0$$

Then assume the thrust line is in the direction of flight.

$$T - D - W\sin\theta = 0$$
$$L - W\cos\theta = 0$$

So the force diagram is,

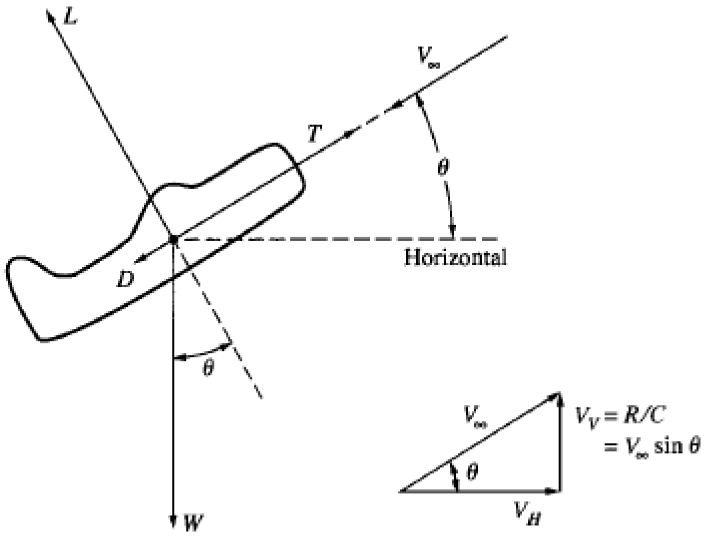


Fig. 2.2. Force and velocity diagrams for climbing flight.

The velocity of the airplane V_∞ is resolved into its horizontal and vertical components V_H and V_V, respectively. The vertical component is, by definition, the rare of climb of the airplane; we denote the rate of climb by R/C. From this diagram,

$$R/C = V_\infty \sin\theta$$

Take the first equation of motion,

$$T - D - W\sin\theta = 0$$
$$\frac{T-D}{W} = \sin\theta$$

When the above multiplied with V_∞,

$$V_\infty \frac{(T-D)}{W} = V_\infty \sin\theta$$
$$\frac{TV_\infty - DV_\infty}{W} = V_\infty \sin\theta = R/C$$

where, TV_∞, is the power available, and DV_∞, is the power required to overcome the drag. So,

$$TV_\infty - DV_\infty = excess\ power$$

Hence,

$$R/C = \frac{excess\ power}{W}$$

Clearly, rate of climb depends on raw power in combination with the weight of the airplane. The higher the thrust, the lower the drag, and the lower the weight, the better the climb performance.

From the second equation of motion,

$$L = W\cos\theta$$

For steady climbing flight, lift is less than weight; indeed, for climbing flight, part of the weight of the airplane is supported by the thrust, and hence less lift is needed than for level flight. In turn, this has an impact on drag; less lift means less drag due to lift. For a given velocity V_∞, the drag in climbing flight is less than that for level flight. From the drag polar,

$$D = q_\infty S C_D = q_\infty S\left(C_{D,0} + KC_L^2\right)$$

where,

$$C_L = \frac{L}{q_\infty S} = \frac{W\cos\theta}{q_\infty S}$$

Then the drag is,

$$D = q_\infty S\left[C_{D,0} + K\left(\frac{W\cos\theta}{q_\infty S}\right)^2\right]$$

$$D = q_\infty S C_{D,0} + \frac{KW^2\cos^2\theta}{q_\infty S}$$

We know that,

$$R/C = V_\infty\sin\theta = V_\infty\frac{(T-D)}{W}$$

$$V_\infty\sin\theta = \frac{V_\infty}{W}\left[T - \left(q_\infty S C_{D,0} + \frac{KW^2\cos^2\theta}{q_\infty S}\right)\right]$$

$$V_\infty\sin\theta = V_\infty\left[\frac{T}{W} - \frac{1}{2}\rho_\infty V_\infty^2\left(\frac{W}{S}\right)^{-1}C_{D,0} - \frac{W}{S}\frac{2K\cos^2\theta}{\rho_\infty V_\infty^2}\right]$$

The above equation is the key to the exact solution of the climb performance of an airplane.

2.4. Gliding (Unpowered) Flight

Whenever an airplane is flying such that the power required is larger than the power available, it will descend rather than climb. In the ultimate situation, there is no power at all; in this case, the airplane will be in gliding, or unpowered, flight. This will occur for a conventional airplane when the engine quits during flight. The force diagram for an unpowered aircraft in descending flight is shown in Fig. 2.3. For steady, unaccelerated descent, where θ is the equilibrium glide angle,

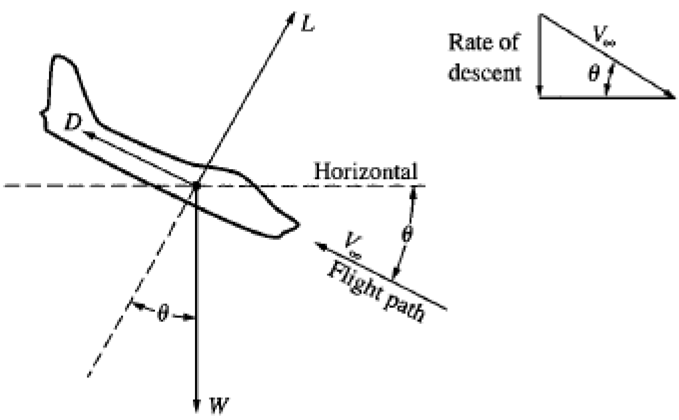


Fig. 2.3. Force and velocity diagrams for gliding Right.

$$L = W\cos\theta$$
$$D = W\sin\theta$$

The equilibrium glide angle is

$$\frac{D}{L} = \frac{\sin\theta}{\cos\theta}$$

$$\tan\theta = \frac{1}{L/D}$$

Clearly, the glide angle is strictly a function of the lift-to-drag ratio; the higher the L/D, the shallower the glide angle. From above equation, the smallest equilibrium glide angle occurs at $(L/D)_{max}$.

$$\tan\theta_{min} = \frac{1}{(L/D)_{max}}$$

For an aircraft at a given altitude h, this is the case for maximum horizontal distance covered over the ground. This distance, denoted by R, is illustrated in Fig. 2.4 for a constant θ.

The equilibrium glide angle θ does not depend on altitude or wing loading, or the like; it simply depends on the lift-to-drag ratio.

However, to achieve a given L/D at a given altitude, the aircraft must fly at a specified velocity V_∞, called the equilibrium glide velocity, and this value of V_∞ does depend on the altitude and wing loading, as follows. Since

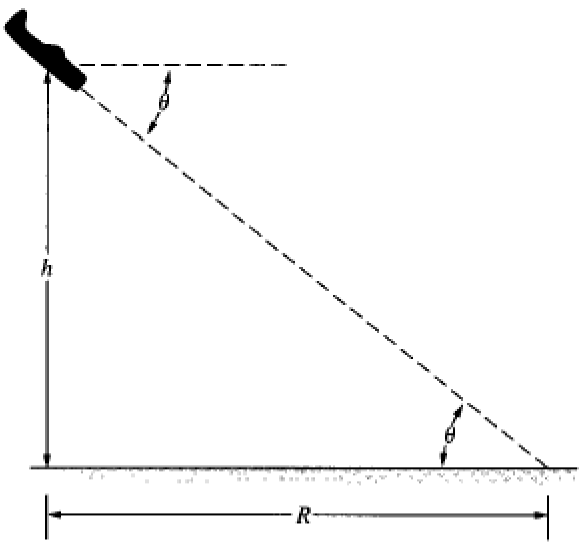


Fig. 2.4. Range covered in on equilibrium guide.

$$L = W\cos\theta$$

$$\frac{1}{2}\rho_\infty V_\infty^2 S C_L = W\cos\theta$$

$$V_\infty = \sqrt{\frac{2\cos\theta}{\rho_\infty C_L}\frac{W}{S}}$$

Clearly, the equilibrium glide velocity depends on altitude and wing loading. The value of C_L in above equation is that particular value which corresponds to the specific value of L/D used in the equilibrium glide angle. Recall that both C_L and L/D are aerodynamic characteristics of the aircraft that vary with angle of attack, as sketched in Fig. 2.5. Note from Fig. 5.4 1 that a specific value of L/D say $(L/D)_1$, corresponds to a specific angle of attack ∞_1, which in turn dictates the lift coefficient $(C_L)_1$,. If L/D is held constant throughout the glide path, then C_L is constant along the glide path. However, the equilibrium velocity along this glide path will change with altitude, decreasing with decreasing altitude.

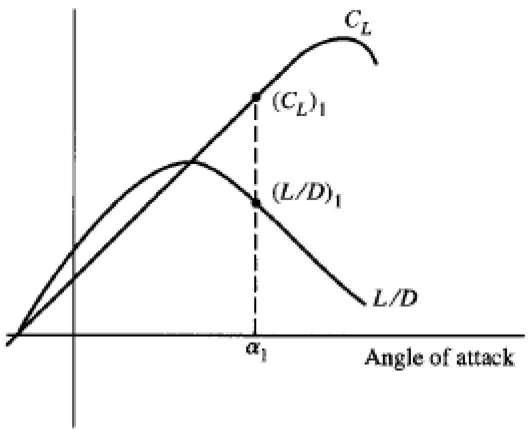


Fig. 2.5. Sketch of the variation of C_L and L/D versus angle of attack for a given airplane.

Consider again the case for a minimum glide angle. For a typical modem airplane, $(L/D)_{max} = 15$, and for this case from minimum glide angle equation $\theta_{min} = 3.8°$ - a small angle. Hence, we can reasonably assume $\cos\theta = 1$ for such cases. Recall,

$$\left(\frac{L}{D}\right)_{max} = \sqrt{\frac{1}{4\,C_{D,0}K}}$$

and for $L = W$, the velocity at which L/D is maximum is given by,

$$V_{(L/D)max} = \left(\frac{2}{\rho_\infty}\sqrt{\frac{K}{C_{D,0}}}\frac{W}{S}\right)^{1/2}$$

Hence, for θ_{min}, the above is the equilibrium velocity along the glide path.

The rate of descent, sometimes called the sink rate, is the downward vertical velocity of the airplane V_V. It is, for unpowered flight, the analog of rate of climb for powered flight. As seen in the insert in Fig. 2.3,

$$Rate\ of\ descent = V_V = V_\infty \sin\theta$$

Rate of descent is a positive number in the downward direction. Multiplying Drag by V_∞, we have

$$DV_\infty = W\sin\theta\, V_\infty = WV_V$$

$$V_V = \frac{DV_\infty}{W}$$

DV_∞, is simply the power required for steady, level flight. Hence, the variation of V_V with velocity is the same as the power required curve, divided by the weight. This variation is sketched in Fig. 2.6, with positive values of V_V, increasing along the downward vertical axis. Clearly, minimum sink rate occurs at the flight velocity for minimum power required. Hence the conditions for minimum sink rate are the same as those for $(P_R)_{min}$,

$$\frac{C_L^{3/2}}{C_D}\ is\ maximum$$

$$V_{\infty_{min.\ sink\,rate}} = \left(\frac{2}{\rho_\infty}\sqrt{\frac{K}{3C_{D,0}}}\frac{W}{S}\right)^{1/2}$$

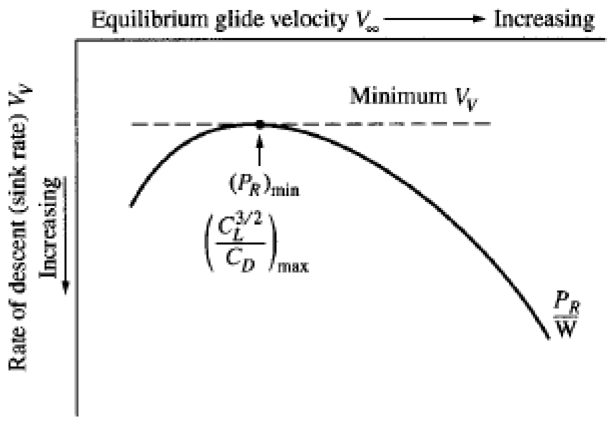


Fig. 2.6. Rate of descent versus equilibrium glide velocity

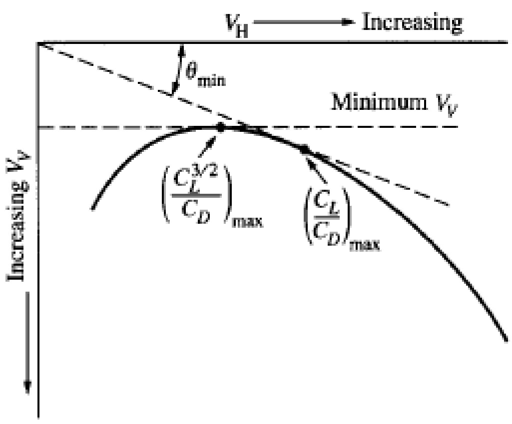


Fig. 2.7. Hodograph for unpowered flight.

The hodograph diagram is sketched in Fig. 2.7 where a line from the origin tangent to the hodograph curve defines θ_{min}. This sketch is shown just to emphasize that the minimum sink rate does not correspond to the minimum glide angle. The flight velocity for the minimum sink rate less than that for minimum glide angle. An analytical expression for the sink rate V_V can be obtained as follows.

$$L = W\cos\theta = \frac{1}{2}\rho_\infty V_\infty^2 S C_L$$

$$V_\infty = \sqrt{\frac{2\cos\theta}{\rho_\infty C_L}\frac{W}{S}}$$

Then,

$$Rate\ of\ descent = V_V = V_\infty \sin\theta = \sin\theta\sqrt{\frac{2\cos\theta}{\rho_\infty C_L}\frac{W}{S}}$$

Dividing drag by lift, we obtain

$$\sin\theta = \frac{D}{L}\cos\theta = \frac{C_D}{C_L}\cos\theta$$

So,

$$V_V = \frac{C_D}{C_L}\cos\theta\sqrt{\frac{2\cos\theta}{\rho_\infty C_L}\frac{W}{S}} = \sqrt{\frac{2\cos^3\theta}{\rho_\infty (C_L^3/C_D^2)}\frac{W}{S}}$$

By making the assumption that $\cos\theta = 1$,

$$V_V = \sqrt{\frac{2}{\rho_\infty (C_L^3/C_D^2)}\frac{W}{S}}$$

Above equation explicitly shows that $(V_V)_{min}$ occurs at $\left(C_L^{3/2}/C_D\right)_{max}$. It also shows that the sink rate decreases with decreasing altitude and increases as the square root of the wing loading.

2.5. Time to Climb

The rate of climb, by definition, is the vertical component of the airplane's velocity, which is simply the time rate of change of altitude dh/dt. Hence,

$$\frac{dh}{dt} = R/C$$

Or

$$dt = \frac{dh}{R/C}$$

In above equation R/C is a function of altitude, and dt is the small increment in time required to climb the small height dh at a given instantaneous altitude. The time to climb from one altitude h_1 to another h_2 is obtained by integrating above equation between the two altitudes:

$$t = \int_{h_1}^{h_2} \frac{dh}{R/C}$$

Normally, the performance characteristic labeled time to climb is considered from sea level, where $h_1 = 0$. Hence, the timc to climb from sea level to any given altitude h_2 is,

$$t = \int_{0}^{h_2} \frac{dh}{R/C}$$

If the maximum rate of climb is used at each altitude, then t becomes the minimum time to climb to altitude h_2.

$$t_{min} = \int_{0}^{h_2} \frac{dh}{(R/C)_{max}}$$

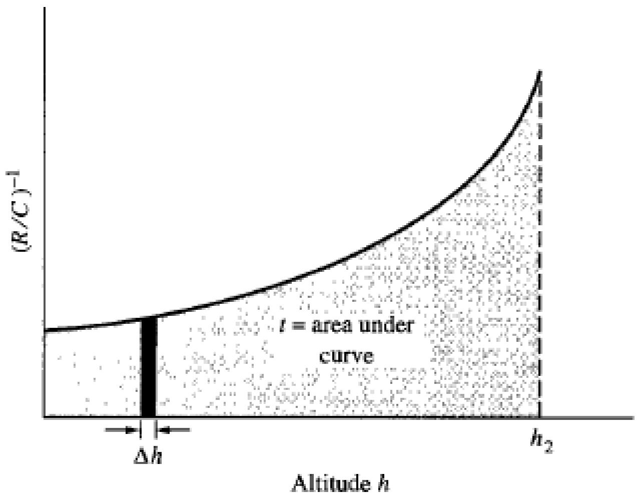


Fig. 2.8. Graphical representation of the time to climb to altitude h_2.

2.6. Level Turn

The flight path and forces for an airplane in a level turn are sketched in Fig. 2.9. Here, the flight path is curved, in contrast to the rectilinear motion studied in steady flight.

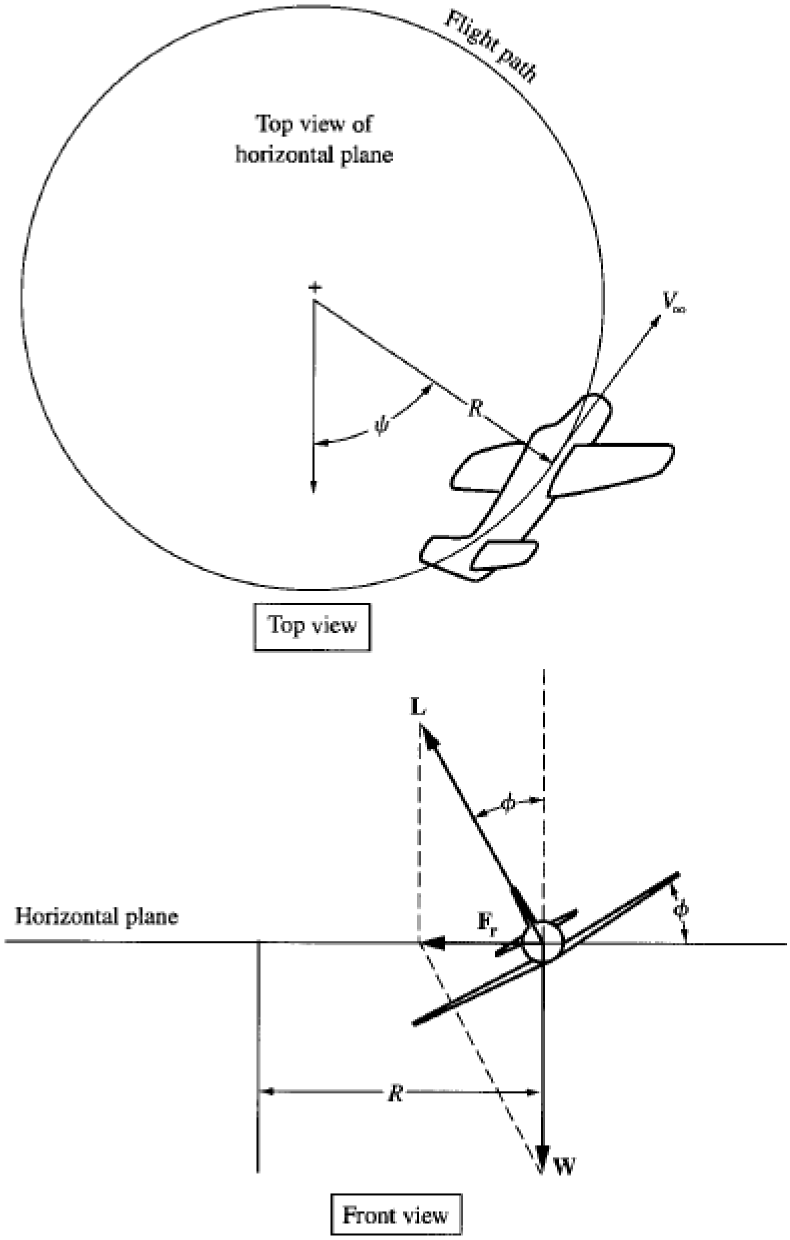


Fig. 2.9. An airplane in a level turn.

By definition, a level turn is one in which the curved flight path is in a horizontal plane parallel to the plane of the ground; that is, in a level turn the altitude remains constant.

The relationship between forces required for a level turn is illustrated in Fig. 2.9. Here, the airplane is banked through the roll angle $\emptyset$. The magnitude of the lift L and the value of $\emptyset$ are adjusted such that the vertical component of lift, denoted by $L \cos \emptyset$, exactly equals the weight,

$$L \cos \emptyset = W$$

Under this condition, the altitude of the airplane will remain constant. Hence, the above equation applies only to the case of a level turn; indeed, it is the necessary condition for a level turn. Another way of stating this necessary condition is to consider the resultant force F_r, which is the vector sum of vectors L and W.

As shown in Fig. 2.9, for the case of the level turn, the magnitude and direction of L are adjusted to be just right so that the vector sum of L and W results in F_r always being in the horizontal plane. In this fashion the altitude remains constant.

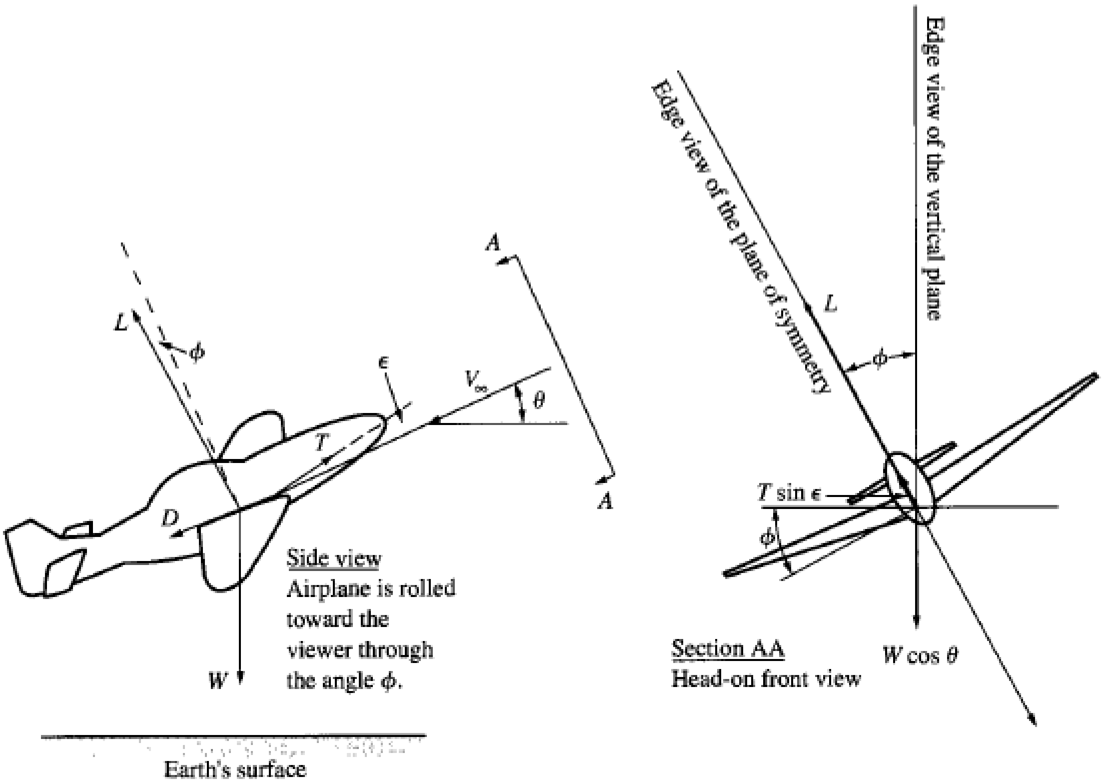


Fig. 2.10. Airplane in climbing flight and rolled through angle ∅.

The generalized force diagram for an airplane in climbing and banking flight is given in Fig. 2.10. When this figure is specialized for level flight, that is, $\theta = 0$, and assuming the thrust vector is parallel to the free-stream direction, that is, $\epsilon = 0$, then the force diagram for a level turn is obtained as sketched in Fig. 2.9. The governing equation of motion is given by

$$m\frac{(V_\infty \cos\theta)^2}{r_2} = L\sin\emptyset + T\sin\epsilon\sin\emptyset$$

The above equation is specialized for the case of $\theta = 0$ and $\epsilon = 0$, namely,

$$m\frac{V_\infty^2}{r_2} = L\sin\emptyset$$

We see that r_2 is the local radius of curvature of the flight path in the horizontal plane. This is the same as the radius R shown in Fig. 2.9. Hence, for a level turn, the governing equation of motion is,

$$m\frac{V_\infty^2}{R} = L\sin\emptyset$$

The above equation is simply a physical statement that the centrifugal force mV_∞^2/R balanced by the radial force $L\sin\emptyset$.

The two performance characteristics of greatest importance in turning flight are

1. The turn radius R.
2. The turn rate $\omega \equiv d\psi/dt$, where ψ is defined in Fig. 2.9. The turn rate is simply the local angular velocity of the airplane along the curved flight path.

These characteristics are particularly germane to combat aircraft. For superior dog-fighting capability, the airplane should have the smallest possible turn radius R and the fastest possible turn rate ω.

The airplane is turning due to the radial force F_r. The larger the magnitude of this force F_r, the tighter and faster will be the turn. The magnitude F_r is the horizontal component of the lift $L \sin \emptyset$. As L increases, F_r increases for two reasons:

1. The length of the lift vector increases.
2. $\emptyset$ increases because for a level turn, $L \cos \emptyset$ must remain constant, namely, equal to W.

Hence, the lift vector L controls the turn; when a pilot goes to turn the airplane, he or she rolls the airplane in order to point the lift vector in the general direction of the turn. Keep in mind that L and $\emptyset$, are not independent; they are related by the condition for a level turn, which can be written as

$$\cos \emptyset = \frac{W}{L} = \frac{1}{L/W}$$

In above equation, the ratio L/W is an important parameter in turning performance; it is defined as the load factor n, hence

$$\cos \emptyset = \frac{1}{n}$$

$$\emptyset = Arccos \frac{1}{n}$$

The roll angle $\emptyset$ depends only on the load factor; if you know the load factor, then you know $\emptyset$, and vice versa. The turn performance of an airplane strongly depends on the load factor.

To obtain an expression for the turn radius, insert $m = W/g$ in the governing equation of motion for level turn, and solve for R.

$$R = \frac{W}{L} \frac{V_\infty^2}{g \sin \emptyset} = \frac{V_\infty^2}{gn \sin \emptyset}$$

and from the trigonometric identity,

$$\sin^2 \emptyset + \cos^2 \emptyset = 1$$

$$\sin^2 \emptyset + \left(\frac{1}{n}\right)^2 = 1$$

$$\sin \emptyset = \sqrt{1 - \frac{1}{n^2}} = \frac{1}{n}\sqrt{n^2 - 1}$$

Then the turn radius is expressed as,

$$R = \frac{V_\infty^2}{g\sqrt{n^2 - 1}}$$

From above equation, the turn radius depends only on V_∞ and n. To obtain the smallest possible R, we want

1. The highest possible load factor (i.e., the highest possible L/W).
2. The lowest possible velocity.

To obtain an expression for the turn rate ω, return to Fig. 2.9 and recall from physics that angular velocity is related to R and V_∞ as

$$\omega = \frac{d\psi}{dt} = \frac{V_\infty}{R}$$

Then we have,

$$\omega = \frac{g\sqrt{n^2 - 1}}{V_\infty}$$

To obtain the largest possible turn rate, we want smallest possible turn radius.

Minimum Turn Radius

The conditions for minimum turn radius R are found by setting $dR/dV_\infty = 0$. The algebra will be simpler if we deal with dynamic pressure, $q = \frac{1}{2}\rho_\infty V_\infty^2$, rather than V_∞. Hence, the turn radius can be written in terms of q_∞,

$$R = \frac{2q_\infty}{g\rho_\infty\sqrt{n^2-1}} \rightarrow (1)$$

Differentiate eqn (1) with respect to q_∞ and setting the derivative equal to zero, we have

$$\frac{dR}{dq_\infty} = \frac{g\rho_\infty\sqrt{n^2-1}\,(2) - 2q_\infty\left(g\rho_\infty(n^2-1)^{-1/2}\cdot n\,\frac{dn}{dq_\infty}\right)}{g^2\rho_\infty^2(n^2-1)} = 0$$

$$2g\rho_\infty\sqrt{n^2-1} - 2g\rho_\infty q_\infty\frac{n}{\sqrt{n^2-1}}\frac{dn}{dq_\infty} = 0$$

multiply by $\sqrt{n^2-1}$ & divide by $2g\rho_\infty$ results,

$$n^2 - 1 - q_\infty n\frac{dn}{dq_\infty} = 0 \rightarrow (2)$$

The load factor n, can be written in terms of q_∞.

$$n^2 = \frac{q_\infty}{K(W/S)}\left(\frac{T}{W} - q_\infty\frac{C_{D,0}}{W/S}\right) \rightarrow (3)$$

Differentiate eqn ③ with respect to q_∞ gives

$$n \frac{dn}{dq_\infty} = \frac{T/W}{2k(W/S)} - \frac{q_\infty C_{D,0}}{k(W/S)^2} \rightarrow ④$$

Substitute eqns ③ & ④ in eqn ②.

$$\frac{q_\infty}{k(W/S)} \frac{T}{W} - \frac{q_\infty^2 C_{D,0}}{k(W/S)^2} - 1 - \frac{q_\infty}{2k(W/S)} \frac{T}{W} + \frac{q_\infty^2 C_{D,0}}{k(W/S)^2} = 0$$

$$\frac{q_\infty}{2k(W/S)} \frac{T}{W} = 1.$$

Then

$$q_\infty = \frac{2k(W/S)}{(T/W)} \rightarrow ⑤$$

Since $q_\infty = \frac{1}{2} \rho_\infty V_\infty^2$, the above eqn becomes,

$$\frac{1}{2} \rho_\infty V_\infty^2 = \frac{2k(W/S)}{(T/W)}$$

$$V_\infty^2 = \frac{4k(W/S)}{\rho_\infty (T/W)}$$

$$(V_\infty)_{R_{min}} = \sqrt{\frac{4k(W/S)}{\rho_\infty (T/W)}} \rightarrow ⑥$$

Equation ⑥ gives the value of V_∞ which corresponds to the minimum turning radius.

The load factor corresponding to this velocity can be found by substituting eqn ⑤ in eqn ③.

$$n^2 = \frac{2k(W/S)(T/W)}{k(W/S)(T/W)} - \frac{4k^2(W/S)^2 C_{D,0}}{(T/W)^2 k (W/S)^2}$$

$$= 2 - \frac{4k\, C_{D,0}}{(T/W)^2}$$

$$\therefore \quad n_{R_{min}} = \sqrt{2 - \frac{4k\, C_{D,0}}{(T/W)^2}} \longrightarrow ⑦$$

Equation ⑦ gives the load factor corresponding to the minimum turning radius.

Finally, the expression for minimum turning radius is obtained by substituting eqns ⑥ & ⑦ in the turn radius eqn.

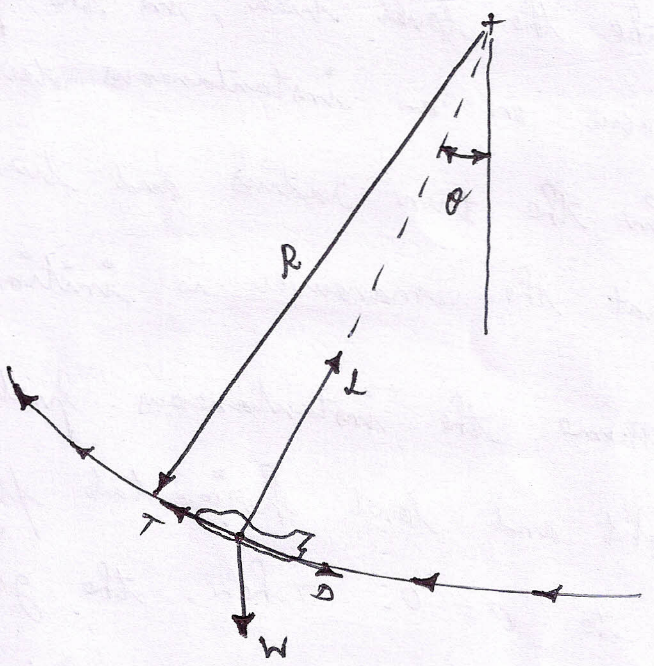
$$R_{min} = \frac{(V_\infty)^2_{R_{min}}}{g\sqrt{n^2_{R_{min}} - 1}} = \frac{4k(W/S)}{g\rho_\infty (T/W)\sqrt{2 - \frac{4k\, C_{D,0}}{(T/W)^2} - 1}}$$

$$\therefore \quad \boxed{R_{min} = \frac{4k(W/S)}{g\rho_\infty (T/W)\sqrt{1 - \frac{4k\, C_{D,0}}{(T/W)^2}}}}$$

Hence, the minimum turn radius is obtained.

The Pull-up and Pulldown Maneuvers.

Consider an airplane initially in straight and level flight, where $L = W$. The pilot suddenly pitches the airplane to a higher angle of attack such that the lift suddenly increases. Because $L > W$, the airplane will arch upward as sketched in figure.



The Pull-up Maneuver.

The flight path becomes curved in the vertical plane, with a turn radius R and turn rate $d\theta/dt$. This is called the pull-up maneuver.

For the pull-up maneuver, the roll angle is zero, $\phi = 0$. This figure is the specialized case of forces projected into the plane formed by the local free-stream velocity V_∞, where $\phi = 0$ and $\epsilon = 0$.

when these applied on Newton's second law, taken perpendicular to the flight path is

$$m \frac{V_\infty^2}{R} = L - W \cos\theta \rightarrow (1)$$

Where r_1 is replaced by R.

This is the governing equation of motion for the flight path.

Unlike the level turn, in the pull-up maneuver we will focus on an instantaneous turn, where we are interested in the turn radius and turn rate at the instant that the maneuver is initiated.

Assume the instantaneous pull-up is initiated from straight and level horizontal flight; this corresponds to $\theta = 0$. Then, the governing equation of motion for the flight path will become.

$$m \frac{V_\infty^2}{R} = L - W.$$

As in the case of the level turn, the pull-up performance characteristics of greatest interest are the turn radius R and turn rate $\omega = \frac{d\theta}{dt}$.

The instantaneous turn radius is obtained from above eqn as follows.

$$R = \frac{mV_\infty^2}{L-W} = \frac{W}{g}\,\frac{V_\infty^2}{L-W} = \frac{V_\infty^2}{g\left(\frac{L}{W}-1\right)}$$

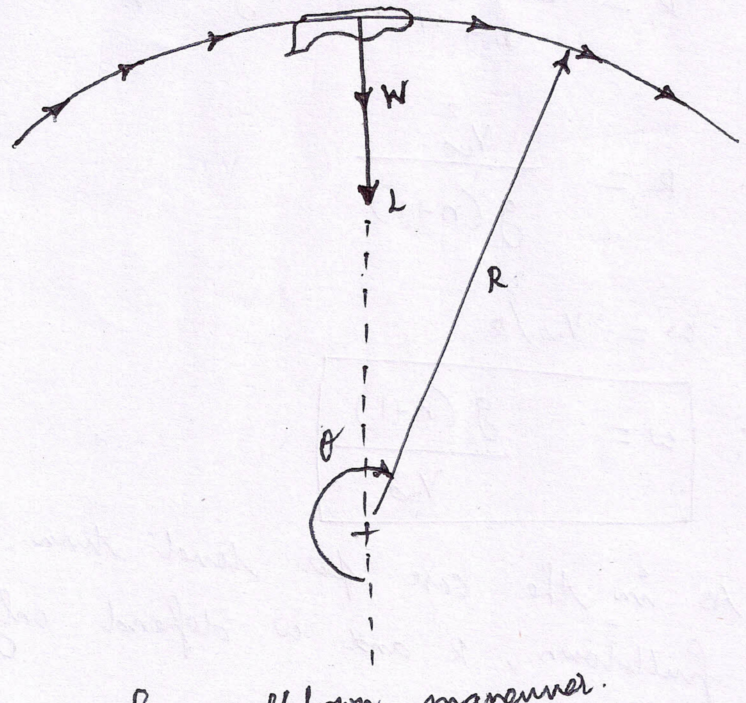
$$\therefore \quad \boxed{R = \frac{V_\infty^2}{g(n-1)}}$$

The instantaneous turn rate is given by

$$\omega = \frac{V_\infty}{R}$$

$$\therefore \quad \boxed{\omega = \frac{g(n-1)}{V_\infty}}$$

A related case is the pulldown maneuver, sketched in the following figure.



The Pulldown maneuver.

Here an airplane initially in straight and level flight is suddenly rolled to an inverted position, such that both L and W are pointing downward.

The airplane will begin to turn downward in a flight path with instantaneous turn radius R and turn rate $\omega = \frac{d\theta}{dt}$.

For this case, the equation of motion is with θ taken as $180°$. For this case, Newton's second law, taken perpendicular to the flight path is,

$$m \frac{V_\infty^2}{R} = L + W$$

Hence,

$$R = \frac{mV_\infty^2}{L+W} = \frac{W}{g} \frac{V_\infty^2}{L+W} = \frac{V_\infty^2}{g\left(\frac{L}{W}+1\right)}$$

$$\therefore \quad R = \frac{V_\infty^2}{g(n+1)}$$

and $\quad \omega = V_\infty / R.$

$$\therefore \quad \boxed{\omega = \frac{g(n+1)}{V_\infty}}$$

As in the case for level turn, for the pull-up and pulldown, R and ω depend only on the flight characteristics V_∞ and n.

2.7. The V-n Diagram

There are two categories of structural limitations on the maximum load factor allowed for a given airplane.

1. *Limit load factor*. This is the boundary associated with permanent structural deformation of one or more parts of the airplane If n is less than the limit load factor, the structure may deflect during a manoeuvre, but it will return to its original state when $n = 1$. If n is greater than the limit load factor, then the airplane structure will experience a permanent deformation, that is, it will incur structural damage.
2. *Ultimate load factor*. This is the boundary associated with outright structural failure. If n is greater than the ultimate load factor, parts of the airplane will break.

Both the aerodynamic and structural limitations for a given airplane are illustrated in the V-n diagram, a plot of load factor versus flight velocity, as given in Fig. 2.11. A V-n diagram is a type of "flight envelope" for a given airplane; it establishes the manoeuvre boundaries. The curve between points A and B represents the aerodynamic limit on load factor imposed by $(C_L)_{max}$, This curve is literally a plot of following equation.

$$n_{max} = \frac{1}{2}\rho_\infty V_\infty^2 \frac{(C_L)_{max}}{W/S}$$

The region above curve AB in the V-n diagram is the stall region. To understand the significance of curve AB better, consider an airplane flying at velocity V_1, where V_1 is shown. Assume the airplane is at an angle of attack such that $C_L < (C_L)_{max}$.

This flight condition is represented by point 1. Now assume the angle of attack is increased to that for $(C_L)_{max}$, keeping the velocity constant at V_1. The lift increases to its maximum value for the given V_1, and hence the local factor $n = L/W$ reaches its maximum value for the given V_1. This value of n_{max} is given by above, and the corresponding flight condition is given by point 2. If the angle of attack is increased further, the wing stalls and the load factor decreases. Therefore, point 3 is unobtainable in flight. Point-3 is in the stall region of the V-n diagram.

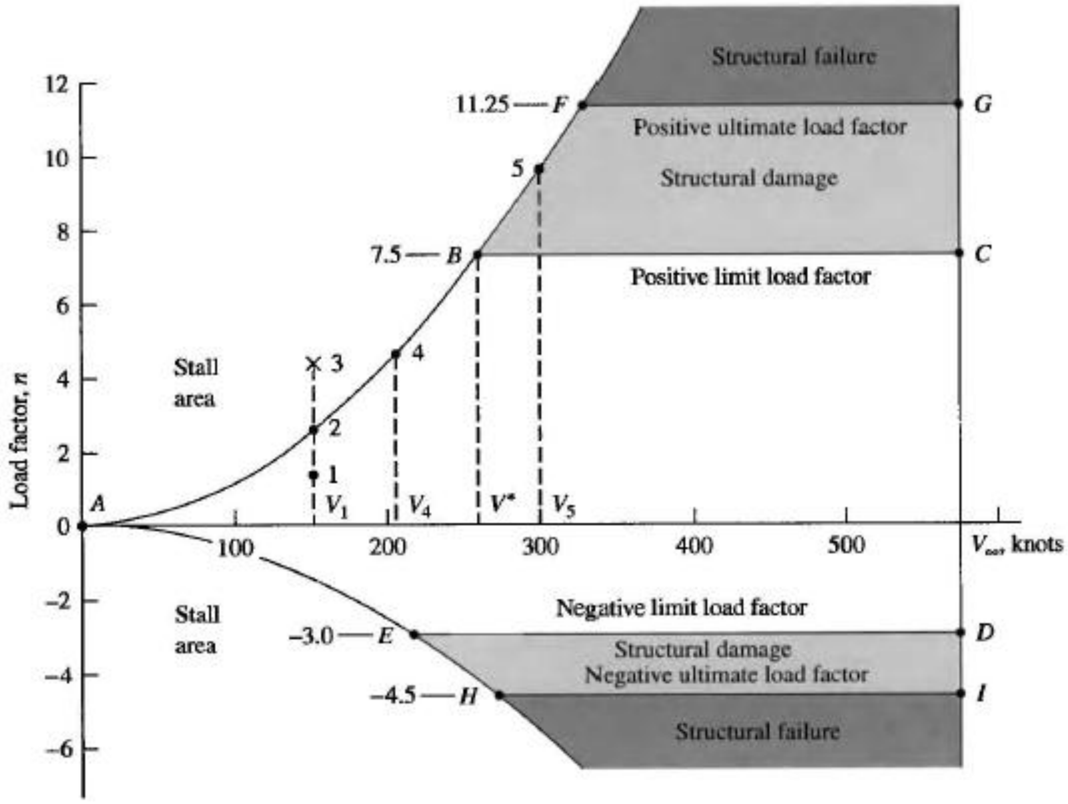


Fig. 2.11. The V-n Diagram

Consequently, point 2 represents the highest possible load factor that can be obtained at the given velocity V_1. As V_∞ is increased, say, to a value of V_4, then the maximum possible load factor n_{max} also increases, as given by point 4. However, n_{max} cannot be allowed to increase indefinitely. It is constrained by the structural limit load factor, given by point B.

The horizontal line BC denotes the positive limit load factor in the V-n diagram. The flight velocity corresponding to B is designated as V^*. At velocities higher than V^*, say, V_5, the airplane must fly at values of C_L less than $(C_L)_{max}$ so that the positive limit load factor is not exceeded. If flight at $(C_L)_{max}$ is obtained at velocity V_5, corresponding to point 5, then structural damage or possibly structural failure will occur. The right-hand side of the V-n diagram, line CD, is a high-speed limit. At flight velocities higher than this limit the dynamic pressure is higher than the design range for the airplane. This will exacerbate the consequences of other undesirable phenomena that may occur in high speed flight, such as encountering a critical gust and experiencing destructive flutter, aileron reversal, wing or surface divergence, and severe compressibility buffeting.

Any one of these phenomena in combination with the high dynamic pressure could cause structural damage or failure. The high-speed limit velocity is a red-line speed for the airplane; it should never be exceeded. By design, it is higher than the level fight maximum cruise velocity V_{max}, by at least a factor of 1.2. It may be as high as the terminal dive velocity of the aircraft. The bottom part of the V-n diagram, given by curve AE and the horizontal line ED corresponds to negative absolute angles of attack, that is, negative lift, and hence the load factors are negative quantities. Curve AE defines the stall limit.

Line ED gives the negative limit load factor, beyond which structural damage will occur. Line HI gives the negative ultimate load factor beyond which structural failure will occur.

For instantaneous manoeuvre performance, point B on the V-n diagram is very important. This point is called the manoeuvre point. At this point, both C_L and n are simultaneously at their highest possible values that can be obtained anywhere throughout the allowable flight envelope of the airplane. In turn, this point is simultaneously corresponds to the smallest possible instantaneous turn radius and the largest possible instantaneous turn rate for the airplane.

The velocity corresponding to point B is called the corner velocity and is designated by V^*. The corner velocity can be obtained by solving Eq. (6.23) for velocity, yielding

$$V^* = \sqrt{\frac{2n_{max}}{\rho_\infty (C_L)_{max}} \frac{W}{S}}$$

In above equation, the value of n_{max} corresponds to that at point B in Fig. 2.11. The corner velocity is an interesting dividing line. At flight velocities less than V^*, it is not possible to structurally damage the airplane due to the generation of too much lift. In contrast, at velocities greater than V^*, lift can be obtained that can structurally damage the aircraft, and the pilot must make certain to avoid such a case.